The Points of (Locally) Compact Regular Formal Topologies

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Abstract

In a paper appeared in 1990, C.J. Mulvey established a constructive characterization of completely prime filters on a compact regular locale L; although proved by intuitionistic logic, the result relies on a notion of maximality which contains an impredicative second-order quantification. In this note we present an alternative concept of maximality, entirely phrased in first-order terms, and give a predicative characterization of the points of a compact regular formal topology (equivalently, we give a characterization of the points of a compact regular locale which can be entirely carried out within Intuitionistic Type Theory). This result is then generalized to locally compact regular formal topologies (resp. locally compact regular locale).

Introduction

Formal Topology[1] was conceived with the aim of developing point-free topology (Locale Theory) in a constructive and predicative foundational setting, such as Martin-Löf's Intuitionistic Type Theory. Quite recently, the topological notion of regularity has been predicatively formulated in this framework, and the class of compact regular formal topologies has shown to have nice and promising properties, particularly from a constructive standpoint (cf. e.g. [5], [3] and [14]). In this note we establish a constructive characterization of the points of a compact regular formal topology, in which formal points are shown to coincide

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[1]Formal Topology was introduced in [15]; a more recent presentation is contained in [16], [17].

with particular subsets of basic neighbourhoods, the *maximal regular* ones. The specific feature of this characterization is that regular subsets will be considered to be maximal according to an entirely first-order criterion of maximality.

This result can then be seen to improve a previous characterization appeared in the context of Locale Theory: in [9] indeed Chris Mulvey introduces a particular formulation of the notion of maximality for regular filters which allows to prove intuitionistically that the completely prime filters on a compact regular locale coincide with the maximal regular filters (cf. [9]). In such a notion of maximality, however, an impredicative second-order quantification appears which makes the result incompatible with foundational settings for constructive mathematics such as Martin-Löf's Intuitionistic Type Theory and Aczel's Constructive Set Theory. A natural relation then exists between Formal Topology and Locale Theory (cf. [15]) that allows to give the following reading of our result: a characterization of completely prime filters on a compact regular locale by means of maximal regular filters can be obtained intuitionistically *and* predicatively, and such a characterization can be entirely carried out within Intuitionistic Type Theory (or Constructive Set Theory).

Few modifications allow to generalize this result to *locally* compact regular formal topologies (and thus to locally compact regular locales). Then, in particular, examples of topologies for which these characterizations are valid are (that giving rise to) the Continuum, Cantor space and the spaces $\mathcal{L}(\mathcal{A})$ of linear functionals of norm ≤ 1 from a semi-normed space A to the reals[2].

1 Preliminaries

We recall the basic definitions of Formal Topology ([15]). The reader is referred to [15], [16] and [17] for a detailed account (the presentation we are to adopt appears in [16]). We use a special notation for subsets, introduced and motivated in [18], which allows to work within Intuitionistic Type Theory (henceforth simply Type Theory, cf. [8], [13]) with essentially the usual mathematical formalism: for the present purpose it will suffice to know that a *subset* U of a set S is a unary predicate (dependent type) on S, $U(x)(x \in S)$, and that a *set-indexed family of subsets* is a binary predicate $U(x,i)(x \in S, i \in I)$ on the sets S and I, where for each $\bar{i}$, $U(x,\bar{i})(x \in S)$ is the subset of index $\bar{i}$ (for simplicity, we will also use the traditional notations $\{a \in S : U(a)\}$, to indicate the subset U, and $U_i(i \in I)$ for a family of subsets). Finally, we will write $a \epsilon U$ to mean $a \in S$ *and* $U(a)$ *true* (in the expression $a \epsilon U$ the symbol 'ϵ' is used, instead of '$\in$', to recall that we are considering a subset, i.e. a propositional function, and not a set; cf. [18]).

1.1 A (formal) topology is a triple $\mathcal{S} \equiv (S, \lhd, \mathsf{Pos})$ where S is a set, called the *base*, $\lhd$ is a relation between elements and subsets of S which satisfies the

[2]Endowed with the *weak** topology, cf. [4].

following conditions:

$$(\textit{reflexivity}) \qquad \frac{a \,\epsilon\, U}{a \lhd U}$$

$$(\textit{transitivity}) \qquad \frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

$$(\downarrow\textit{-right}) \qquad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow V}$$

where

$$\begin{array}{rcl} U \lhd V & \equiv & (\forall u \,\epsilon\, U) u \lhd V \\ U \downarrow V & \equiv & \{d : S |\ (\exists u \,\epsilon\, U)\ (d \lhd \{u\})\ \&\ (\exists v \,\epsilon\, V)\ (d \lhd \{v\})\} \end{array}$$

and Pos is a subset of S which satisfies

$$(\textit{monotonicity}) \qquad \frac{\mathsf{Pos}(a) \quad a \lhd U}{(\exists b \,\epsilon\, U)\mathsf{Pos}(b)}$$

$$(\textit{positivity}) \qquad \frac{\mathsf{Pos}(a) \to a \lhd U}{a \lhd U}.$$

We will write $\mathsf{Pos}(U)$ for $(\exists a \,\epsilon\, U)\mathsf{Pos}(a)$. The relation $\lhd$ is called *cover* and Pos *positivity predicate* (we pronounce $a \lhd U$ as 'U covers a', and $\mathsf{Pos}(a)$ as 'a is positive'). For simplicity, we often loosely confuse singleton subsets $\{a\}$ with elements a.

One can think intuitively of the elements of S as of indexes for the basic neighbourhoods of a topological space; the cover relation can then be seen as a formal description of the inclusion between basic neighborhoods and subsets of S, and the predicate Pos as a positive way to express that a certain neighbourhood is non-empty. Then, for instance, 'monotonicity' has the following intuitive reading: if a non-empty neighbourhood is covered by a family of neighbourhoods (indexed by U) then there must be at least one member of the family which is non-empty.

An equivalent formulation of positivity (cf. [15]) which we will use in the following is

$$\frac{a \lhd U}{a \lhd U^+},$$

where $U^+ \equiv \{b \,\epsilon\, U : \mathsf{Pos}(b)\}$ (that is, only non-empty neighbourhoods contribute to the cover). [3]

[3]Notice that one has $\neg\mathsf{Pos}(U) \iff U =_S \emptyset$, while $\mathsf{Pos}(U) \iff U \neq_S \emptyset$ cannot be obtained constructively in general (cf. [15]). It has to be remarked that the positivity predicate is never really needed in the following.

1.2 In a formal topology $\mathcal{S}$ a *formal point* is a subset $\alpha \subseteq S$ such that

$i.$ $(\exists a)(a \,\epsilon\, \alpha)$ $\qquad$ $ii.$ $(a \,\epsilon\, \alpha \;\&\; b \,\epsilon\, \alpha) \to (\exists c)(c \,\epsilon\, a \downarrow b \;\&\; c \,\epsilon\, \alpha)$

$iii.$ $\dfrac{a \,\epsilon\, \alpha \quad a \lhd U}{(\exists b)(b \epsilon U \;\&\; b \,\epsilon\, \alpha)}$ $\qquad$ $iv.$ $a \,\epsilon\, \alpha \to \mathsf{Pos}(a).$

We will denote by $\mathsf{Pt}(\mathcal{S})$ the collection of formal points. (Condition $iv.$ is actually known to be derivable from $iii.$ and positivity and could thus be skipped[4]).

Finally, we recall that given two subsets U, V of S, $U =_{\mathcal{S}} V$ means exactly that $U \lhd V \;\&\; V \lhd U$, and that for $U \subseteq S$, the *(pseudo-)complement* U^* of U is given by $U^* \equiv \{b : \neg\mathsf{Pos}(b \downarrow U)\}$.

1.3 The relation with Locale Theory can be sketched as follows (a detailed discussion of this subject is contained in [15], [17]): defining, for $U \subseteq S$, $\mathcal{S}U$ to be the subset $\{a \in S : a \lhd U\}$, we say that U is *saturated* if $U = \mathcal{S}U$; denoting then by $Sat(\mathcal{S})$ the collection of saturated subset of $\mathcal{S}$, $Sat(\mathcal{S})$ endowed with the operations

$$\mathcal{S}U \wedge \mathcal{S}V \equiv \mathcal{S}U \cap \mathcal{S}V = \mathcal{S}(U \downarrow V)$$

and

$$\bigvee_{i \in I} \mathcal{S}U_i \equiv \mathcal{S}(\bigcup_{i \in I} U_i)$$

forms a *frame* (or locale, or complete Heyting algebra).

From a non-constructive point of view the converse is also valid (that is, any frame can be obtained as the frame of saturated subsets of a formal topology $\mathcal{S}$). Finally, the points of a formal topology $\mathcal{S}$ are easily shown to correspond to completely prime filters on $Sat(\mathcal{S})$.

1.4 A formal topology $\mathcal{S} \equiv (S, \lhd, \mathsf{Pos})$ is said to be *compact* if whenever $S \lhd U$ there exists a finite[5] subset $U_0 \subseteq U$ such that $S \lhd U_0$.

The notion of regularity have been recently introduced in Formal Topology as the predicative counterpart of that given in the context of locales (see for instance [6]): for a, b in S, we say that b is *well-covered* by a iff $S \lhd a \cup b^*$; defining $wc(a)$ to be the subset of neighbourhoods b which are well-covered by a, $wc(a) \equiv \{b : S \lhd a \cup b^*\}$, a formal topology $\mathcal{S}$ is then said to be *regular* if for all a in S, $a \lhd wc(a)$[6]. A topology $\mathcal{S}$ will be said to be *compact regular* if it is compact and regular.

[4] A proof is recalled in [12]. A generalized definition of Formal Topology can however be considered in which the positivity rule is not required, cf. [16].

[5] Note that here and in the following a set, or a subset, is considered to be 'finite' if its elements can be listed; cf. the notions of finite and sub-finite in [2].

[6] This definition appeared in [14]; in case of compactness, it is equivalent to the one introduced in [5], [3].

The following lemmas will be used in the following, often without an explicit mention; they obtain in any formal topology $\mathcal{S}$.

Lemma 1.5. *Let V, W, Z be subsets of S, and $U_i (i \in I)$ be a family of subsets of S. We have*

i) $V \cup (W \downarrow Z) =_{\mathcal{S}} (V \cup W) \downarrow (V \cup Z)$,

ii) $(\bigcup_i U_i) \downarrow V = \bigcup_i (U_i \downarrow V)$.

Proof. i) Let $a \,\epsilon\, V \cup (W \downarrow Z)$, i.e. $(a \,\epsilon\, V \vee a \,\epsilon\, (W \downarrow Z))$. If $a \,\epsilon\, V$, then trivially $a \,\epsilon\, (V \cup W) \downarrow (V \cup Z)$; if $a \,\epsilon\, (W \downarrow Z)$ then $(\exists b \,\epsilon\, W)(\exists c \,\epsilon\, Z)(a \lhd b \;\&\; a \lhd c)$, whence $(\exists b \,\epsilon\, W \cup V)(\exists c \,\epsilon\, Z \cup V)(a \lhd b \;\&\; a \lhd c)$, i.e. $a \,\epsilon\, (V \cup W) \downarrow (V \cup Z)$. Conversely, let $a \,\epsilon\, (V \cup W) \downarrow (V \cup Z)$; by definition, there is $b \,\epsilon\, (V \cup W)$ and $c \,\epsilon\, (V \cup Z)$ such that $a \lhd b$ and $a \lhd c$. From $b \,\epsilon\, (V \cup W)$ and $c \,\epsilon\, (V \cup Z)$ we have that $b \,\epsilon\, V$ or $b \,\epsilon\, W$, and $c \,\epsilon\, V$ or $c \,\epsilon\, Z$. It is then straightforward to see that a is covered by $V \cup (W \downarrow Z)$.

ii) By definition, $(\bigcup_i U_i) \downarrow V \equiv \{a : (\exists b)(\exists c)(b \,\epsilon\, \bigcup_i U_i \;\&\; c \,\epsilon\, V \;\&\; a \lhd b \;\&\; a \lhd c)\} = \{a : (\exists b)(\exists c)((\exists i \in I) b \,\epsilon\, U_i \;\&\; c \,\epsilon\, V \;\&\; a \lhd b \;\&\; a \lhd c)\} = \{a : (\exists i \in I)(\exists b)(\exists c)(b \,\epsilon\, U_i \;\&\; c \,\epsilon\, V \;\&\; a \lhd b \;\&\; a \lhd c))\} = \bigcup_i (U_i \downarrow V)$.

Lemma 1.6. *Let b and V be respectively an element and a subset of S; if $S \lhd b^* \cup V$, then $b \lhd V$.*

Proof. From $S \lhd b^* \cup V$ we have $b \lhd b^* \cup V$; since $b \lhd b$, we have (by $\downarrow -right$) that $b \lhd (b^* \cup V) \downarrow b$. Then $b \lhd (b^* \downarrow b) \cup (V \downarrow b)$, and hence (by positivity) $b \lhd (V \downarrow b)^+ \lhd V$.

Lemma 1.7. *Let $U_i (i \in I)$ be a family of subsets of S. Then $(\bigcup_i U_i)^* = \bigcap_i (U_i^*)$; in particular, for $U, V \subseteq S$, $(U \cup V)^* = U^* \cap V^* = U^* \downarrow V^*$.*

Proof. We have that: $a \,\epsilon\, (\bigcup_i U_i)^* \iff \neg\mathsf{Pos}(a \downarrow \bigcup_i U_i) \iff \neg\mathsf{Pos}(\bigcup_i (a \downarrow U_i)) \iff \neg\mathsf{Pos}(\{b : (\exists i) b \,\epsilon\, (a \downarrow U_i)\}) \iff \neg(\exists c)(c \,\epsilon\, \{b : (\exists i) b \,\epsilon\, (a \downarrow U_i)\} \;\&\; \mathsf{Pos}(c)) \iff \neg(\exists c)((\exists i) c \,\epsilon\, (a \downarrow U_i) \;\&\; \mathsf{Pos}(c)) \iff (\forall i)\neg(\exists c)(c \,\epsilon\, (a \downarrow U_i) \;\&\; \mathsf{Pos}(c)) \iff (\forall i)\neg\mathsf{Pos}(a \downarrow U_i) \iff a \,\epsilon\, \{d : (\forall i \in I) d \,\epsilon\, U_i^*\} \iff a \,\epsilon\, \bigcap_i (U_i^*)$. Moreover, for $U, V \subseteq S$, $(U^* \cap V^*) = (U^* \downarrow V^*)$: indeed clearly $(U^* \cap V^*) \subseteq (U^* \downarrow V^*)$; let then $a \,\epsilon\, (U^* \downarrow V^*)$, there are then $b \,\epsilon\, U^*$ and $c \,\epsilon\, V^*$ such that $a \lhd b$ and $a \lhd c$. Thus (since $b \,\epsilon\, U^* \equiv \neg\mathsf{Pos}(b \downarrow U)$), we have that $\neg\mathsf{Pos}(a \downarrow U)$, and similarly $\neg\mathsf{Pos}(a \downarrow V)$; hence $a \,\epsilon\, (U^* \cap V^*)$.

2 Maximal Regular Points

In [9] a characterization of completely prime filters on a compact regular locale L is obtained by means of intuitionistic logic: completely prime filters are proved to coincide with maximal regular filters, where a regular filter P is said to be *maximal* if and only if it is proper and *for all regular filter* F, if P is contained in F, then

$$a \in F \to (a \in P \vee 0 \in F),$$

for all $a \in L$. The impredicative second-order quantification on filters appearing in this definition cannot be accepted in the context of a constructive and predicative foundational setting: in particular such a characterization could not be formalized within Intuitionistic Type Theory ([7], [13]), nor assuming Peter Aczel's Constructive Set Theory ([1]).

In [9] it is also pointed out that the strength to prove the characterization intuitionistically is provided by the first-order quantification on elements appearing in the definition of maximal regular filter; we are then going to prove that an entirely first-order notion of maximality can be formulated which suffices to prove such a characterization in a completely constructive way. The result is worked out in the context of Formal Topology, thus yielding a proof which can automatically be formalized within Martin-Löf's Type Theory; recalling however the natural relation between Formal Topology and Locale Theory (cf. [15]), it will be easily recognized that the same arguments could as well be phrased in the context of Locale Theory.

2.1 Let $\mathcal{S} \equiv (S, \lhd, \mathsf{Pos})$ be a formal topology. We define a subset α of S to be *regular* if it satisfies

1. $(\exists a)(a \in \alpha)$
2. $(a \in \alpha \;\&\; b \in \alpha) \leftrightarrow (\exists c \in S)(c \in a \downarrow b \;\&\; c \in \alpha)$
3. $a \in \alpha \to \mathsf{Pos}(a)$
4. $a \in \alpha \to (\exists b \in S)(b \in wc(a) \;\&\; b \in \alpha)$

We say that α is a *maximal regular subset* of S[7] if moreover, for all a, b in S,

$$5. \quad b \in wc(a) \to ((\exists c \in S)(c \in \alpha \;\&\; \neg Pos(b \downarrow c)) \vee a \in \alpha)$$

(that is, if a neighbourhood b is well-covered by a neighbourhood a, either a is a neighbourhood of α, or there exists a neighbourhood c of α disjoint from b).

Observe that maximal regular subsets are maximal (between regular subsets) in the usual sense: let $\mathcal{S}$ be any formal topology and let $\alpha \subseteq S$ be a maximal

[7] The idea for defining maximality for regular subsets in this way was inspired by the notion of *maximal approximation*, as formulated in [7].

regular subset. If $\beta \subseteq S$ is a regular subset and $\alpha \subseteq \beta$ we have $\alpha = \beta$: indeed, let $a \,\epsilon\, \beta$; by regularity of β there is $b \,\epsilon\, \beta$ such that $b \,\epsilon\, wc(a)$. Then (by maximality) either there is $c \,\epsilon\, \alpha$ such that $\neg Pos(b \downarrow c)$, or $a \,\epsilon\, \alpha$; but $\alpha \subseteq \beta$ implies that $\mathsf{Pos}(b \downarrow c)$ for all $c \,\epsilon\, \alpha$, hence $a \,\epsilon\, \alpha$.

On a compact locale a regular filter is prime if and only if it is completely prime (cf. e.g. [6], pg. 135). The following lemma expresses this property in our context.

Lemma 2.2. *Let $\mathcal{S}$ be a compact formal topology and let α be a regular subset of $\mathcal{S}$. If U is any subset of S, a is an element of α and $a \lhd U$, then there is a finite subset V_0, $V_0 \subseteq U$, and a basic neighbourhood b in α, such that $b \lhd V_0$.*

Proof. Let U be any subset of S, $a \,\epsilon\, \alpha$ and $a \lhd U$. By regularity of α there is $b \,\epsilon\, wc(a)$, $b \,\epsilon\, \alpha$. Then $S \lhd b^* \cup a$, whence also $S \lhd b^* \cup U$. By compactness, there is a finite U_0, $U_0 \subseteq b^* \cup U$ such that $S \lhd U_0$. Since $U_0 \subseteq b^* \cup U$ means that $U_0(a)$ *true* implies $b^*(a) \vee U(a)$, we have $U_0 = V_0 \cup W_0$, with V_0 finite subset of U and W_0 finite subset of b^*. Thus, by lemma 1.6, $b \lhd V_0$.

By a simple modification of the corresponding argument in [9] we have:

Lemma 2.3. *Let $\mathcal{S}$ be a compact regular formal topology and let α be a regular subset of $\mathcal{S}$. If $a \,\epsilon\, \alpha$ and $a \lhd \{a_1, ..., a_t\}$, there exist $b \,\epsilon\, \alpha$ and finite subsets $V_1, ..., V_t$ such that $V_1 \subseteq wc(a_1), ..., V_t \subseteq wc(a_t)$ and $b \lhd V_1 \cup ... \cup V_t$.*

Proof. Suppose that $a \,\epsilon\, \alpha$ and $a \lhd \{a_1, a_2, ..., a_t\}$. By regularity of α, we can find b in α such that $b \,\epsilon\, wc(a)$, i.e. such that $S \lhd b^* \cup a$. By $a \lhd \{a_1, a_2, ..., a_t\}$ we then have $S \lhd b^* \cup \{a_1, a_2, ..., a_t\}$, which, by regularity of $\mathcal{S}$, gives $S \lhd b^* \cup wc(a_1) \cup ... \cup wc(a_t)$. Compactness then allows to extract a finite subset U_0 of $b^* \cup wc(a_1) \cup wc(a_2) \cup ... \cup wc(a_t)$ such that $S \lhd U_0$. Thus $S \lhd b^* \cup V_1 \cup V_2 \cup ... \cup V_t$, where, for $i = 1, ..., t$, V_i is a finite subset of $wc(a_i)$; by lemma 1.6, we then obtain $b \lhd V_1 \cup ... \cup V_t$, as desired.

In a regular topology points form a sub-collection of the collection of maximal regular subsets:

Proposition 2.4. *Let $\mathcal{S}$ be a regular formal topology. If α is a formal point of $\mathcal{S}$, α is also a maximal regular subset of $\mathcal{S}$.*

Proof. Let $\alpha \in \mathsf{Pt}(\mathcal{S})$ be a formal point. Then α clearly satisfies *1., 2.* and *3.* of 2.1 (to prove that $(\exists c \in S)(c \,\epsilon\, a \downarrow b \;\&\; c \,\epsilon\, \alpha)$ implies $a \,\epsilon\, \alpha$ and $b \,\epsilon\, \alpha$, simply observe that $c \epsilon a \downarrow b$ means $c \lhd a$ and $c \lhd b$, then use condition *iii.* of 1.2). Since $\mathcal{S}$ is regular, we have $a \lhd wc(a)$ for all a, hence (by *iii.* of 1.2) we have that also *4.* is satisfied. To prove maximality, let a, b such that $b \epsilon wc(a)$, that is, such that $S \lhd b^* \cup a$. Let d be a neighbourhood of α (which exists by *i.* of 1.2); by $S \lhd b^* \cup a$, we have $d \lhd b^* \cup a$, and thus (by *iii.* of 1.2) we obtain $(\exists c \epsilon (b^* \cup a))(c \epsilon \alpha)$. This precisely means that there is $c \epsilon \alpha$ such that $\neg\mathsf{Pos}(c \downarrow b)$, or $a \epsilon \alpha$, as required.

We can now prove our main result: if a topology is compact and regular, we have that also the converse of proposition 2.4 is true:

Theorem 2.5. *For any compact regular formal topology $\mathcal{S}$, the formal points of $\mathcal{S}$ are precisely the maximal regular subsets of $\mathcal{S}$.*

Proof. By the preceding proposition, formal points of $\mathcal{S}$ are maximal regular subsets of $\mathcal{S}$. To prove the converse, let $\alpha \subseteq S$ be maximal regular. Obviously *i.*, *ii.* and *iv.* of 1.2 are satisfied. We have then to prove that from $a \epsilon \alpha$ and $a \lhd U$ we can deduce $(\exists b \epsilon U)(b \epsilon \alpha)$. By lemma 2.2, it is enough to prove the claim for finite U. Let then $U = \{a_1, ..., a_t\}$. By lemma 2.3, we can find t finite subsets $V_1, ..., V_t$ of S and a basic neighbourhood b of α such that, for $i = 1, ..., t$, $V_i \subseteq wc(a_i)$ and $b \lhd V_1 \cup ... \cup V_t$. Let

$$V_1 = \{v_1^1, ..., v_{k_1}^1\}, ..., V_t = \{v_1^t, ..., v_{k_t}^t\}.$$

To fix ideas, let us consider V_1: by maximality of α we have that

$$(a_1 \epsilon \alpha \vee (\exists d_1 \epsilon \alpha)\neg\mathsf{Pos}(v_1^1 \downarrow d_1))\&...\&(a_1 \epsilon \alpha \vee (\exists d_{k_1} \epsilon \alpha)\neg\mathsf{Pos}(v_{k_1}^1 \downarrow d_{k_1})),$$

which yields

$$a_1 \epsilon \alpha \vee ((\exists d_1 \epsilon \alpha)\neg\mathsf{Pos}(v_1^1 \downarrow d_1)\&...\&(\exists d_{k_1} \epsilon \alpha)\neg\mathsf{Pos}(v_{k_1}^1 \downarrow d_{k_1})).$$

If $d_i \epsilon \alpha$, for $i = 1, ..., k_1$, then, by *2.* of 2.1, there is $e^1 \epsilon \alpha$, with $e^1 \epsilon d_1 \downarrow d_2 \downarrow ... \downarrow d_{k_1}$. It follows that $a_1 \epsilon \alpha \vee (\exists e^1 \epsilon \alpha)\neg\mathsf{Pos}(e^1 \downarrow V_1)$. Similarly we obtain $a_j \epsilon \alpha \vee (\exists e^j \epsilon \alpha)\neg\mathsf{Pos}(e^j \downarrow V_j)$, for $j = 2, ..., t$. We can thus deduce that

$$a_1 \epsilon \alpha \vee ... \vee a_t \epsilon \alpha \vee ((\exists e^1 \epsilon \alpha)\neg\mathsf{Pos}(e^1 \downarrow V_1)\&...\&(\exists e^t \epsilon \alpha)\neg\mathsf{Pos}(e^t \downarrow V_t)).$$

Suppose now that $(\exists e^1 \epsilon \alpha)\neg\mathsf{Pos}(e^1 \downarrow V_1)\&...\&(\exists e^t \epsilon \alpha)\neg\mathsf{Pos}(e^t \downarrow V_t)$. Thus, since b and e^j belong to α, by *2.* of 2.1. we have that $(\exists f \epsilon \alpha)$ $(f \epsilon (b \downarrow e^1 \downarrow ... \downarrow e^t))$, whence, by *3.* of 2.1, $\mathsf{Pos}(b \downarrow e^1 \downarrow ... \downarrow e^t)$; from $b \lhd V_1 \cup ... \cup V_t$ we obtain $(b \downarrow e^1 \downarrow ... \downarrow e^t) \lhd V_1 \cup ... \cup V_t$. Then, using monotonicity, $\mathsf{Pos}((b \downarrow e^1 \downarrow ... \downarrow e^t) \downarrow (V_1 \cup ... \cup V_t))$. However, for $i = 1, ..., t$, also $\neg\mathsf{Pos}((b \downarrow e^1 \downarrow ... \downarrow e^t) \downarrow V_i)$ (since $\neg\mathsf{Pos}(e^i \downarrow V_i)$), whence $\neg\mathsf{Pos}((b \downarrow e^1 \downarrow ... \downarrow e^t) \downarrow (V_1 \cup ... \cup V_t))$, which gives a contradiction. Thus $a_1 \epsilon \alpha \vee ... \vee a_t \epsilon \alpha$, as desired.

Theorem 2.5 thus authorizes to speak of maximal regular *points* of a compact regular formal topology. A relevant example of a compact regular formal topology is that giving rise to the space of linear functionals of norm ≤ 1 from a semi-normed space A to the reals[8]: in [4] the topology is proved to be compact; following the ideas in [11] regularity can also be proved.

3 A generalization: Locally Compact Regular Formal Topologies

In this section we show that the characterization we have obtained can be extended with a few modifications to the wider class of locally compact regular formal topologies; an instance of such topologies is the (topology giving rise to the) continuum (or, more in general, the n-dimensional Euclidean space)[9].

3.1 We essentially follow [12] by defining a formal topology $\mathcal{S}$ to be *locally compact* if there is an indexed family $i(a)(a \in S)$ of subsets of S such that for all $a \in S$ and $U \subseteq S$,

$$a \lhd U \iff (\forall b \,\epsilon\, i(a))(b \lhd U_b),$$

where U_b is a finite subset (depending on b) of U. In [12] it is shown that if a topology $\mathcal{S}$ satisfies such a condition, we also have $i(a) =_{\mathcal{S}} a$.

We say that a topology is *locally compact regular* if it is locally compact and regular.

The following lemma states a relation between the subsets $wc(a)$ and $i(a)$ in a locally compact regular topology.

Lemma 3.2. *If $\mathcal{S}$ is a locally compact regular topology, $i(a) \subseteq wc(a)$ for all a in S. Hence, a locally compact formal topology is regular if and only if $i(a) \subseteq wc(a)$ for all a.*

Proof. Let $b \,\epsilon\, i(a)$, then, since $a \lhd wc(a)$ there is a finite subset $U_b \equiv \{c_1, c_2 ..., c_k\}$ of $wc(a)$ such that $b \lhd \{c_1, c_2 ..., c_k\}$; to prove that $b \,\epsilon\, wc(a)$, observe that $S \lhd (a \cup c_i^*)$, for $i = 1, ..., k$, implies that $S \lhd (a \cup c_1^*) \downarrow (a \cup c_2^*) \downarrow ... \downarrow (a \cup c_k^*)$; by lemma 1.5 we thus have $S \lhd a \cup (c_1^* \downarrow c_2^* \downarrow ... \downarrow c_k^*)$, and, by lemma 1.7, we

[8]The space $\mathcal{L}(\mathcal{A})$ was presented as a locale in [10], [11]; it has been introduced via a formal topology in [4]: linear functionals are obtained as formal points of a formal topology whose base is intuitively given by a base of the *weak* topology*.

[9]See e.g. [4] for the definition of the topology $\mathcal{R}$ used to introduce the continuum.

obtain $S \lhd a \cup (c_1 \cup ... \cup c_k)^*$. It is then enough to show that $(c_1 \cup ... \cup c_k)^* \subseteq b^*$: let $d \,\epsilon\, (c_1 \cup ... \cup c_k)^*$, i.e. $\neg\mathsf{Pos}(d \downarrow (c_1 \cup ... \cup c_k))$; if we had $\mathsf{Pos}(d \downarrow b)$, we would also have $\mathsf{Pos}(d \downarrow (c_1 \cup ... \cup c_k))$ (since $b \lhd \{c_1, c_2 ..., c_k\}$), contradiction. Hence $\neg\mathsf{Pos}(d \downarrow b)$, i.e. $d \,\epsilon\, b^*$; thus $(c_1 \cup ... \cup c_k)^* \subseteq b^*$, as required.

Compact regular and locally compact regular topologies are in the expected convenient relation:

Proposition 3.3. *If $\mathcal{S}$ is compact regular, $\mathcal{S}$ is locally compact regular with $i(a) \equiv wc(a)$, for all $(a \in S)$.*

Proof. $\mathcal{S}$ compact regular means that $S \lhd U \to S \lhd U_0$ for U_0 finite subset of U. Define $i(a)(a \in S)$ to be the family $wc(a)(a \in S)$: if $a \lhd U$ and $b \,\epsilon\, wc(a)$, we have $S \;\lhd b^* \cup a \;\lhd\; b^* \cup U$. By compactness and lemma 1.6, there is then a finite U_0, $U_0 \subseteq U$ such that $b \lhd U_0$; the other implication (that is $(\forall b \,\epsilon\, wc(a))(b \lhd U_b) \to a \lhd U$, with U_b finite subset of U depending on b) is obtained recalling that (by regularity) for all a, $a \lhd wc(a)$.

3.4 Let $\mathcal{S}$ be a locally compact formal topology. We define $\alpha \subseteq S$ to be a *continuous subset* of $\mathcal{S}$[10] if it satisfies conditions *1., 2. ,3.* of 2.1 and, instead of *4.*,

4'. $a \,\epsilon\, \alpha \to (\exists b)(b \,\epsilon\, i(a) \;\&\; b \,\epsilon\, \alpha)$

We say that α is a *maximal continuous subset* of $\mathcal{S}$ if moreover, for all a, b in S

5'. $b \,\epsilon\, i(a) \to ((\exists c \in S)(c \,\epsilon\, \alpha \;\&\; \neg Pos(b \downarrow c)) \vee a \,\epsilon\, \alpha)$

As one can expect, if $\mathcal{S}$ is a locally compact formal topology, α and β are respectively a maximal continuous subset and a continuous subset of $\mathcal{S}$, we have that $\alpha \subseteq \beta$ implies $\alpha = \beta$ (argue as in the preceding section, simply replacing $wc(a)$ with $i(a)$).

We now show that results analogous to lemmas 2.2 and 2.3 obtain for locally compact formal topologies (observe that regularity is not required in lemma 3.6).

Lemma 3.5. *Let $\mathcal{S}$ be a locally compact topology, and let α be a continuous subset of S. If $a \,\epsilon\, \alpha$, U is any subset of S and $a \lhd U$, then a neighbourhood b of α can be found such that $b \lhd U_b$, where U_b is a finite subset of U.*

[10] The relation, established in [12], between (what we here call) locally compact formal topologies and distributive continuous lattices has motivated this terminology.

Proof. From $a \;\epsilon\; \alpha$ and *4'.*, we obtain an element b such that $b \;\epsilon\; \alpha$ and $b \;\epsilon\; i(a)$; by local compactness we have that $b \lhd U_b$, where U_b is a finite subset of U.

Lemma 3.6. *Let $\mathcal{S}$ be a locally compact formal topology, and let α be a continuous subset of S. If $a \;\epsilon\; \alpha$ and $a \lhd \{a_1, ..., a_t\}$, finite subsets $V_1, ..., V_t$ and a neighbourhood b of α can be found such that, for $k = 1, ..., t$, $V_k \subseteq i(a_k)$ and $b \lhd V_1 \cup ... \cup V_t$.*

Proof. Since α is continuous, there is b in α, $b \;\epsilon\; i(a)$; since $a_k \lhd i(a_k)$ for $k = 1, ..., t$, by local compactness we have $b \lhd U_0$, with U_0 finite subset of $i(a_1) \cup ... \cup i(a_t)$. Thus $b \lhd V_1 \cup ... \cup V_t$, with, for $k = 1, ..., t$, V_k are finite subsets of $i(a_k)$.

Lemmas 3.5 and 3.6 allow to prove that:

Proposition 3.7. *In a locally compact formal topology $\mathcal{S}$ a maximal continuous subset of $\mathcal{S}$ is a formal point of $\mathcal{S}$.*

Proof. Let α be a maximal continuous subset. The only non-obvious fact to prove is that, from $a \;\epsilon\; \alpha$ and $a \lhd U$, we can deduce $(\exists b \;\epsilon\; U)(b \;\epsilon\; \alpha)$. By lemma 3.5 we may consider only finite U. Let then $U = \{a_1, ..., a_t\}$, $a \;\epsilon\; \alpha$ and $a \lhd \{a_1, ..., a_t\}$, we want to prove that $a_1 \;\epsilon\; \alpha \vee ... \vee a_t \;\epsilon\; \alpha$. By lemma 3.6, we can find finite subsets $V_1, ..., V_t$ and $b \;\epsilon\; \alpha$ such that, for $k = 1, ..., t$, $V_k \subseteq i(a_k)$, and $b \lhd V_1 \cup ... \cup V_t$. The proof then proceeds in complete analogy with that of theorem 2.5.

Now a characterization of the points of locally compact regular formal topologies corresponding to the one proved in the previous section for compact regular topologies can be obtained:

Theorem 3.8. *For any locally compact regular formal topology $\mathcal{S}$, the formal points of $\mathcal{S}$ are precisely the maximal continuous subsets of $\mathcal{S}$.*

Proof. By the preceding proposition we only need to show that if $\alpha \in \mathsf{Pt}(\mathcal{S})$ is a formal point, then it is a maximal continuous subset of $\mathcal{S}$. We have already showed (proposition 2.4) that a formal point α satisfies *1., 2., 3.*. Since $\mathcal{S}$ is locally compact, we have $a \lhd i(a)$ for all a, and thus by *iii)* of 1.2 we have that also *4'.* is satisfied. To prove *5'.*, let a, b such that $b \;\epsilon\; i(a)$; by lemma 3.2, $b \;\epsilon\; wc(a)$, i.e. $S \lhd b^* \cup a$. Then we may argue exactly as in the proof of proposition 2.4 to conclude that there is $c \;\epsilon\; \alpha$ such that $\neg \mathsf{Pos}(c \downarrow b)$, or $a \;\epsilon\; \alpha$.

Observe that the characterization we have proved in this section is a proper generalization of that obtained for compact regular topologies: indeed, by proposition 3.3, we could have derived the characterization provided in 2.5 by theorem 3.8.

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References

[1] Aczel, Peter, "The type-theoretic interpretation of constructive set theory". Logic Colloquium '77 (Proc. Conf., Wroclaw, 1977), pp. 55–66, *Stud. Logic Foundations Math.*, 96 (North-Holland), 1978.

[2] Bishop, Errett, and Bridges, Douglas, *Constructive Analysis* (Springer-Verlag), Berlin, 1985.

[3] Cederquist, Jan, and Coquand, Thierry, "Entailment relations and distributive lattices". Logic Colloquium '98 (Prague), 127–139, *Lecture Notes Logic*, 13, Assoc. Symbol. Logic, Urbana, IL, 2000.

[4] Cederquist, Jan, Coquand, Thierry, and Negri, Sara, "The Hahn-Banach theorem in Type Theory". In: *Twenty-Five years of Constructive Type Theory*, Proceedings of a Congress held in Venice, October 1995, G. Sambin and J. Smith eds. (Oxford University Press), 1998, pp. 57-72.

[5] Coquand, Thierry, "Formal Compact Hausdorff Spaces", unpublished.

[6] Johnstone, Peter T., *Stone Spaces*, Cambridge University Press, 1982.

[7] Martin-Löf, Per, *Notes on Constructive Mathematics* (Almqvist and Wiksell) Stockholm, 1970.

[8] Martin-Löf, Per, *Intuitionistic Type Theory*. Studies in Proof Theory. Lecture Notes, 1. (Bibliopolis), Napoli, 1984.

[9] Mulvey, Chris J., "The maximality of filters". Special issue in honor of B. Banaschewski. *Journal of Pure and Applied Algebra* 68 (1990), no. 1-2, 253–258.

[10] Mulvey, Chris J., and Pelletier, Joan W., "The dual locale of a seminormed space". *Cahiers de Topologie et Geometrie Differentielle*, vol. 23 (1987), n.1, pp.73-92.

[11] Mulvey, Chris J., and Pelletier, Joan W., "A globalization of the Hahn-Banach theorem". *Advances in Mathematics* 89 (1991), pp. 1-60.

[12] Negri, Sara, "Continuous lattices in formal topology, 1997". In *Lecture Notes in Computer Science*, E. Gimenez and C. Paulin (eds), selected papers of the Types meeting, Aussois, 1996, Lecture Notes in Computer Science 1512, pp. 333-353, Springer, 1998.

[13] Nordström, Bengt, Petersson, Kent, and Smith, Jan, *Programming in Martin-Löf's Type Theory* (Clarendon Press), Oxford, 1990.

[14] Sambin, Giovanni, A monograph on Formal Topology, unpublished manuscript.

[15] Sambin, Giovanni, "Intuitionistic formal spaces - a first communication". In *Mathematical Logic and its Applications*, D. Skordev, ed. Plenum, New York - London 1987, 187-204

[16] Sambin, Giovanni, "Formal topology and domains". Proceedings of the Workshop on Domains IV, Informatik-Bericht Nr. 99-01, Universität-GH Siegen, 1999.

[17] Sambin, Giovanni, and Gebellato, Silvia, "A preview of the basic picture: a new perspective on formal topology". In *Types for Proofs and Programs. International Workshop, TYPES '98, Kloster Irsee, Germany, March 27-31, 1998, selected papers.* Lecture Notes in Computer Science, 1657, Springer, eds. T. Altenkirch and W. Naraschewski and B. Reus, 1999, pp. 194-207.

[18] Sambin, Giovanni, and Valentini, Silvio, "Building up a toolbox for Martin-Löf's type theory: subset theory". In: *Twenty-Five years of Constructive Type Theory.* Proceedings of a Congress held in Venice, October 1995, G. Sambin and J. Smith eds., Oxford University Press, 1998, pp. 221-244.